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LARGE ISOTROPIC ELASTIC DEFORMATION OF COMPOSITES AND POROUS MEDIA

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Abstract—The problem of large isotropic deformation of composite materials and porous media consisting of a finitely-deforming elastic matrix and spherical inclusions or voids is analyzed exactly on the basis of the composite spheres assemblage model. For porous media, the stress-strain relations for expansion and contraction are found to be fundamentally different. It is shown that a very small amount of pores has significant effect on expansion, but not on contraction.

1. INTRODUCTION

The theory of linear elastic behavior of composite materials is today a considerably developed subject and many fundamental general and specific results are available. For a recent review see [1]. By contrast, the case of finite elastic behavior, when the strains are not small, is virtually unexplored, because of the enormous mathematic difficulties involved. Important general discussion of macrovariables and macromechanical behavior for composite materials subjected to large strains has been given in [2] and has been summarized in [3]. Some of the pertinent concepts will be outlined below. Results for bulk modulus of composites consisting of second order elastic matrix and a dilute distribution of linear elastic spherical inclusions or voids have been given in [3].

The present work is concerned primarily with a restricted aspect of theory of finite elastic behavior: the isotropic finite expansion or contraction of composites and porous media. This problem will be solved exactly for a special distribution of spherical particles or voids without restriction of volume fraction. The same model has recently been employed in an ad hoc fashion, [4], to study nonlinear viscoelastic compression of porous media.

First, we briefly recall the definition of linear effective elastic properties of composite materials. A large statistically homogeneous composite material body is subjected to boundary displacements

$$u_i(S) = \epsilon^0_{ij} x_j, \tag{1.1}$$

where ϵ^0_{ij} are constant strains and x_j are surface coordinates, or to boundary tractions

$$T_i = \sigma^0_{ij} n_j, \tag{1.2}$$

where σ^0_{ij} are constant stresses and n_j are the components of the surface normal. In homogeneous elastic bodies such boundary conditions produce homogeneous states of stress and strain. In heterogeneous bodies, the local strains and stresses are highly nonhomogeneous. However, the average strain and stress theorems (see, e.g. [5]) assert that when (1.1) is prescribed

$$\bar{\epsilon}_{ij} = \epsilon^0_{ij}, \tag{1.3}$$

and when (1.2) is prescribed

$$\bar{\sigma}_{ij} = \sigma^0_{ij}, \tag{1.4}$$

where overbars denote volume averages, which, by statistical homogeneity, are also averages over representative volume elements (RVE) containing a sufficiently large number of heterogeneities. It follows from linearity that in case (1.1)

$$\bar{\sigma}_{ij} = C^*_{ijkl}\bar{\epsilon}_{kl} \tag{1.5}$$

and in case (1.2)

$$\bar{\epsilon}_{ij} = S^*_{ijkl}\bar{\sigma}_{kl}, \tag{1.6}$$

where **C*** and **S*** are the effective elastic moduli and compliance tensors respectively, which are mutually reciprocal. When the body is statistically isotropic, (1.5) reduces to

$$\bar{\sigma}_{ij} = \lambda^*\bar{\epsilon}_{kk}\delta_{ij} + 2G^*\bar{\epsilon}_{ij}. \tag{1.7}$$

In preparation for finite deformation, we consider first the case of a homogeneous finite elastic body. Defining initial coordinates X_i and current coordinates x_i, we impose on the surface a linear deformation defined by

$$x_i(S) = \lambda_{ij}X_j(S), \tag{1.8}$$

where λ_{ij} are symmetric constants and S is the initial surface. Then the deformation throughout is given by

$$x_i = \lambda_{ij}X_j. \tag{1.9}$$

Dually, the surface deformation could be referred to the deformed surface s. Thus,

$$X_i(s) = \mu_{ij}x_j(s), \tag{1.10}$$

where $\boldsymbol{\mu}$ is the inverse of $\boldsymbol{\lambda}$ and the internal deformation is now given by

$$X_i = \mu_{ij}x_j. \tag{1.11}$$

Obviously, the deformation gradients associated with (1.9) and (1.11) are λ_{ij} and μ_{ij}, respectively.

The stresses are found by the usual finite elasticity stress-strain relations in terms of the strain energy function W (see, e.g. [6]), which is a function of the three invariants of λ_{ij} or μ_{ij} in the isotropic case. Thus the stresses are uniform throughout.

Guided by these results, we impose (1.8) or (1.10) throughout the large deformation process of a composite material body. It may then be shown in terms of the extended divergence theorem, just as in the case of the average small strain theorem, that the volume average gradients are

$$\overline{\frac{\partial x_i}{\partial X_j}} = \lambda_{ij} \qquad \overline{\frac{\partial X_i}{\partial x_j}} = \mu_{ij}, \tag{1.12}$$

the first average being taken over initial volume V and the second over final volume v. These results merely require displacement continuity at constituent interfaces. By contrast, averages of finite strain are much more complex, since such strains are non-linear functions of deformation gradients, and therefore a useful finite strain average theorem does not seem available.

The effective stress-strain relation will be defined as a relation between average stress and average deformation gradient, [2]. It is most convenient to consider the average of the stress field referred to current coordinates, thus the Cauchy stress. This average stress is a function of μ_{ij} and of the deformed internal geometry which defines the macroscopic mechanical behavior. It must be assumed that the composite material remains statistically homogeneous during the deformation process. We may define an RVE in the deformed body just as in the linear case, and all deformation gradient and Cauchy stress averages over *RVE* are then the same as body averages.

Alternatively, the boundary condition (1.2) may be imposed in reference to the deformed surface. Since Cauchy stress satisfies the usual equilibrium equations and usual traction continuity on deformed phase interfaces the average stress theorem remains valid and therefore average Cauchy stress is again given by (1.4). Now the average deformation gradient in current coordinates must be evaluated and its functional dependence on σ_{ij}^0 defines the effective stress-strain relation. The stress strain relations by the two procedures must be the same for statistically homogeneous materials for physical reasons.

Since the average virtual work theorem remains valid in the deformed body, [2], it follows for either one of boundary conditions (1.10) or (1.2) that

$$\overline{\frac{\partial X_i}{\partial x_j}\sigma_{ij}} = \overline{\frac{\partial X_i}{\partial x_j}}\,\bar{\sigma}_{ij},$$

and on this basis it may be shown the elastic energy density W^* is defined either in terms of μ_{ij} or σ_{ij}^0. If this energy is evaluated, the stress-strain relation can be defined in terms of derivatives of W^* with respect to μ_{ij} or σ_{ij}^0[2].

2. COMPOSITE SPHERES ASSEMBLAGE MODEL

The surface of a finite elastic compressible, homogeneous, and isotropic body is deformed so that

$$x_i(S) = \lambda X_i(S), \tag{2.1}$$

where λ is a constant of any magnitude. This boundary condition is a special case of (1.8). Then throughout the body

$$\begin{aligned} x_i &= \lambda X_i \\ u_i(\mathbf{X}) &= (\lambda - 1)X_i, \end{aligned} \tag{2.2}$$

where u_i are the displacements. This is an isotropic shape preserving deformation, and all lengths in all directions change by the factor λ. Since the body is elastic and isotropic, there exists a strain energy function $W(I_1, I_2, I_3)$, where I_1, I_2, I_3 are strain invariants associated here with the Lagrangian strain tensor. In the present case, the stress is isotropic and is given by[6],

$$\begin{aligned} \sigma_{ij} &= \sigma\delta_{ij} \\ \sigma &= 2\left(\frac{1}{\lambda}\frac{\partial W}{\partial I_1} + 2\lambda\frac{\partial W}{\partial I_2} + \lambda^3\frac{\partial W}{\partial I_3}\right) \end{aligned} \tag{2.3}$$

where, for the present isotropic deformation,

$$I_1 = 3\lambda^2 \qquad I_2 = 3\lambda^4 \qquad I_3 = \lambda^6. \tag{2.4}$$

Consider a spherical surface of radius R_2 in the undeformed body. The center of the sphere located at X_1^0, X_2^0, X_3^0 is the origin of an initial coordinate system Y_1, Y_2, Y_3. Then

$$X_i = X_i^0 + Y_i. \tag{2.5}$$

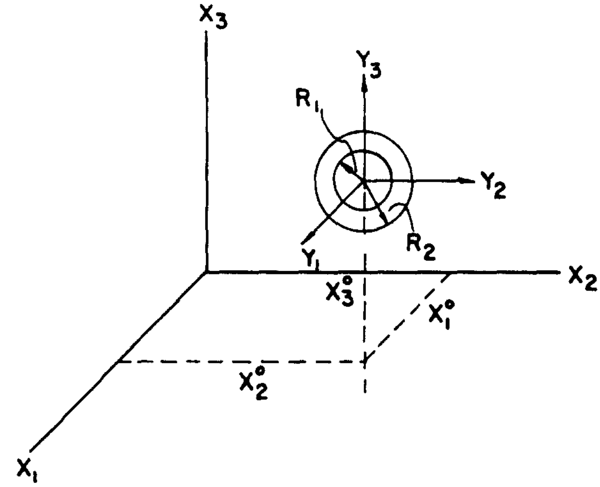

Fig. 1. Coordinates and sphere replacement.

Introducing (2.5) into (2.2), it follows that the center of the sphere moves to the new position $\lambda \mathbf{X}^0$, while the new position of points in and on the sphere relative to the new center is given by

$$y_i = \lambda Y_i. \tag{2.6}$$

The displacement of the center is a rigid body motion which produces no strain or stress. The remaining deformation (2.6) is purely radial and may be written

$$r = \lambda R \quad \text{interior} \tag{2.7a}$$

$$r_2 = \lambda R_2 \quad \text{surface} \tag{2.7b}$$

where R and r are initial and final radii, respectively.

Since the stress tensor is given by (2.3), the traction on r is radial and can be written as

$$\sigma_{rr} = \sigma \tag{2.8}$$

which is defined by the as yet unspecified stress-strain relation of the material.

It is our purpose to replace this homogeneous sphere by a composite or hollow sphere without perturbing the state of deformation and stress in the remainder of the body. A necessary and sufficient condition for this is that if the surface of the heterogeneous sphere of radius R_2 is radially deformed according to (2.7b), then the radial stress necessary to do this will be precisely (2.8). Now, if we pick a composite sphere made of an inner spherical core of radius R_1 and an outer shell with radii R_1, R_2 made of any finitely deforming isotropic materials, as shown in Fig. 1, and impose on its surface the deformation (2.7b), solution of this problem will yield the surface stress $\sigma_{rr}(\lambda)$. We now assign to the homogeneous body with which we commenced the discussion a stress-strain relation (2.3) which will yield the same $\sigma = \sigma_{rr}(\lambda)$. Then, the homogeneous sphere can be replaced by the composite sphere without any perturbation of the remaining part of the body. Now, all composite spheres with same initial ratio R_1/R_2, same constituent materials, but *different* radii R_2 require the same $\sigma_{rr}(\lambda)$ on their surface to produce the surface deformation (2.7b). In the sequel, such spheres will be called similar. It follows that *any* nonoverlapping spherical parts of the homogeneous body can be replaced by similar composite spheres without perturbing the remaining part, as seen in Fig. 2.

Let the deformed volume be v. A part, v_s, of this volume is filled with radially deformed similar composite spheres with surface deformations (2.7b) and the remaining volume is v'. The same configuration is achieved if the initial undeformed body of

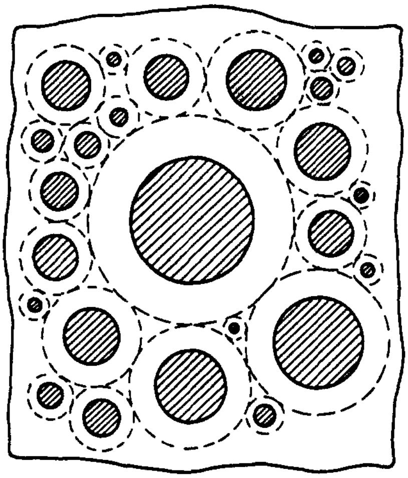

Fig. 2. Composite spheres assemblage.

volume V is partially filled with undeformed composite spheres of total volume V_s and the remaining volume is V', whereupon this heterogeneous body is subjected to the boundary deformation (2.1). Then, any undeformed composite sphere whose center is located at $\mathbf{X}^0$ undergoes a rigid body motion $\lambda\mathbf{X}^0$ and the surface deformation (2.7b) to move into its deformed position.

At this stage, the model is a three phase material consisting of the two phases of the composite spheres and the uniform phase throughout V' or v'. It follows from (2.1) and the average deformation gradient theorem that

$$\overline{\frac{\partial x_i}{\partial X_j}} = \lambda\delta_{ij} \qquad \overline{\frac{\partial X_i}{\partial x_j}} = \frac{1}{\lambda}\,\delta_{ij}. \tag{2.9}$$

Writing (2.8) in the form

$$T_i = \sigma n_i, \tag{2.10}$$

it follows from the average stress theorem that the average stress in any composite sphere is

$$\bar{\sigma}_{ij} = \sigma\delta_{ij}. \tag{2.11}$$

Since in the remaining volume $\sigma_{ij} = \sigma\delta_{ij}$, by construction, it follows that (2.11) is the average stress in the three phase material, and thus the stress-strain relation for isotropic deformation is defined by the relation between σ and λ. Now the remaining volume can be filled out by similar composite spheres of diminishing sizes to any degree of accuracy and thus V' or v' may be made as small as we please, approaching the geometrical arrangement known as the *composite spheres assemblage* which has been introduced in the linear elastic context in [7]. It follows that the stress-strain relation of the model is the $\sigma(\lambda)$ relation of any one composite sphere. It is also clear that the model is not restricted to elasticity, but can be used for any isotropic phases, linear or nonlinear. Pertinent examples are[8] for elastoplastic deformation and[4] for nonlinear viscoelastic deformation.

3. EVALUATION FOR THE FINITE ELASTIC MATRIX

It is assumed that the matrix is described by an incompressible Mooney-Rivlin material through the strain energy density representation

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{3.1}$$

where C_1 and C_2 are constant. This is widely used for rubber-like materials.

We first consider the case of a porous elastic material and, accordingly, we need the solution for a hollow elastic sphere with initial radii R_1, R_2 which expands or contracts isotropically so that

$$r_2 = \lambda R_2. \tag{3.2}$$

The solution to this problem may be found in [6]. The radial stress on r_2 is given by

$$\sigma = \sigma_{rr}(r_2) = C_1[1/\lambda^4 + 4/\lambda - (1/\lambda_1^4 + 4/\lambda_1)] + 2C_2[1/\lambda^2 - 2\lambda - (1/\lambda_1^2 - 2\lambda_1)] \tag{3.3}$$

$$\lambda_1 = r_1/R_1. \tag{3.4}$$

It follows from the incompressibility of the matrix shell material that

$$r_1^3 - R_1^3 = r_2^3 - R_2^3. \tag{3.5}$$

Defining the initial volume concentration of pores by c_0, we have for a single sphere, as well as for the entire assemblage

$$c_0 = R_1^3/R_2^3. \tag{3.6}$$

It follows from (3.5–3.6) that

$$\lambda_1^3 = (\lambda^3 - 1)/c_0 + 1. \tag{3.7}$$

The *apparent* volume dilatation of a sphere as well as of the assemblage is defined by

$$\epsilon_v = (r_2^3 - R_2^3)/R_2^3 = \lambda^3 - 1. \tag{3.8}$$

Thus, (3.7) assumes the form

$$1/\lambda_1^3 = 1 - \epsilon_v/(1 + \epsilon_v)c_0. \tag{3.9}$$

Equation (3.3) with (3.7) defines the effective isotropic stress-strain relation of the assemblage in terms of λ, while (3.3) with (3.8)–(3.9) defines the relation between applied isotropic stress and apparent volume change. It is seen that, to close the pores by pressure $r_1 = 0$, $\lambda_1 = 0$ and the required pressure becomes infinite from (3.3).

It is thus seen that a porous material with an incompressible elastic matrix exhibits macroscopic compressibility or extensibility because of the radial deformations permitted by the pores. Figure 3 shows the effective isotropic stress-strain relations for various initial pore volume fractions when the matrix is a Mooney-Rivlin material with constants $C_1 = 0.5$ MPa, $C_2 = 0.05$ MPa. Such low values are typical for rubbers[9]. The abscissa indicates values of unit length change $\epsilon = \lambda - 1$ or values of apparent volume dilatation ϵ_v. It is seen that the stress-strain relations in tension and compression are very different. The tensile stress-strain relations rise initially, but quickly flatten out and even go down slightly, to rise again. By contrast, the compressive stress strain relations descend steeply and asymptotically to approach the vertical lines $\epsilon_v = c_0$ or $\lambda = (1 + c_0)^{1/3}$ (3.8). With decreasing c_0, the compressive stress-strain relations approach the ordinate $\epsilon = \lambda - 1 = 0$. For this reason, compressive stress-strain relations are shown in larger detail in Fig. 4.

The case of small c_0 is of particular interest. Since the matrix has been assumed incompressible, there is no isotropic strain for any isotropic stress in the case $c_0 = 0$, and thus the stress-strain relation in this case is the ordinate $\epsilon = 0$. It is seen that, for very small c_0, the compressive stress-strain relation approaches this situation, but the tensile one does not. Even for as small a value as $c_0 = 0.001$, significant isotropic

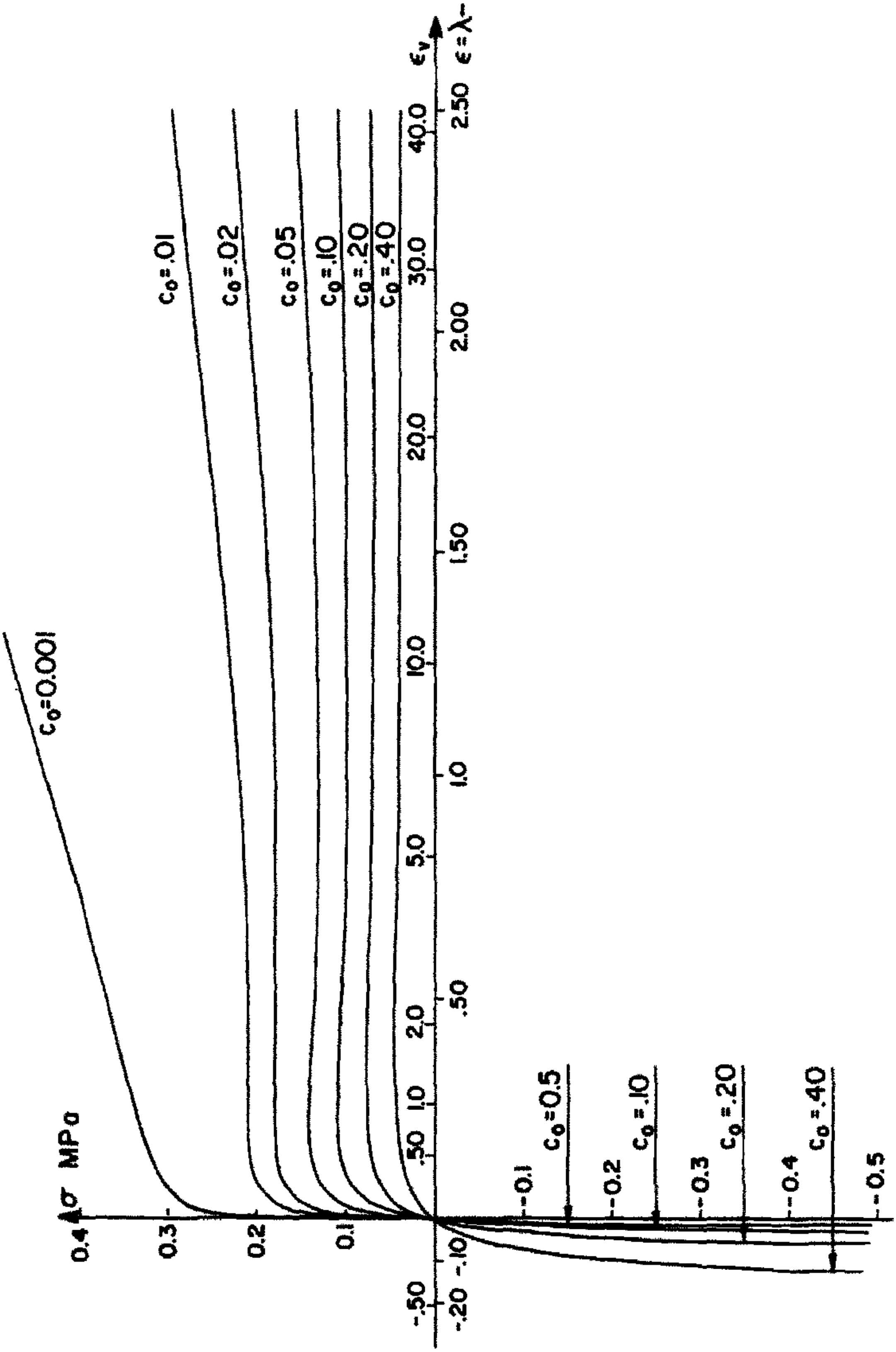

Fig. 3. Stress-strain relations for porous material.

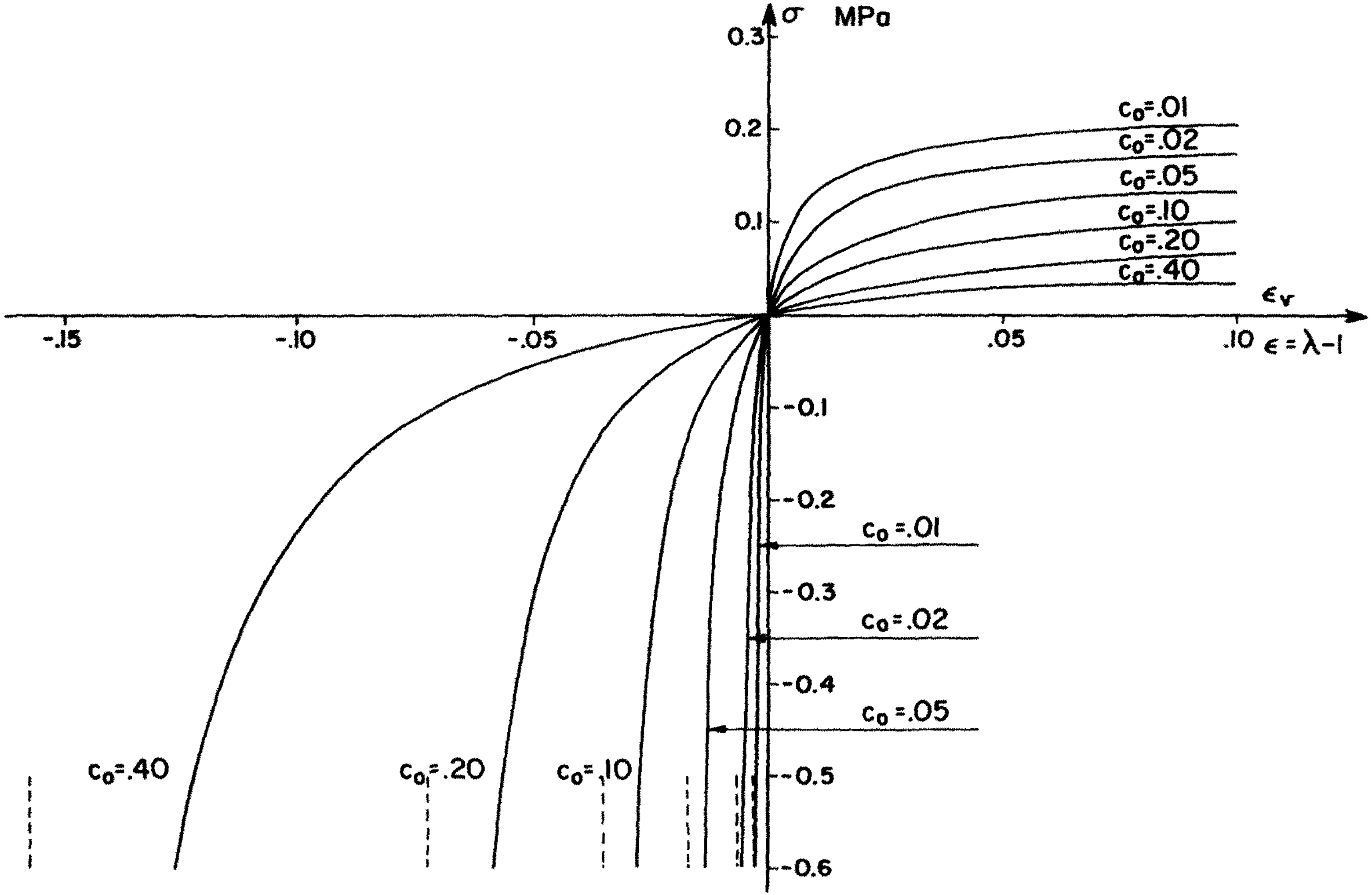

Fig. 4. Compressive stress-strain relations for porous material.

extensibility develops at stretches of about $\epsilon = 0.10$. Since a material will always contain some voids, this seems to imply that finite elastic isotropic inextensibility does not exist in nature. This, however, is a built-in feature of the Mooney-Rivlin material, and therefore the question arises whether such a representation is adequate for large tensile deformations?

Next, we will consider the case of linear elastic inclusions embedded in a Mooney-Rivlin matrix. In this case the model consists of composite spheres where the inner core is linear elastic with bulk modulus K_1 and the matrix shell is finite elastic. There now develops a radial stress σ_1 at the interface $r = r_1$. The radial stress at $r = r_2$ is now given by[6],

$$\sigma = \sigma_{rr}(r_2) = C_1[1/\lambda^4 + 4/\lambda - (1/\lambda_1^4 + 4/\lambda_1)] + C_2[1/\lambda^2 - 2\lambda - (1/\lambda_1^2 - 2\lambda_1)] + \sigma_1. \quad (3.10)$$

Since the inclusion is linear elastic and its deformation is purely radial, it is in a state of infinitesimal isotropic strain and stress. Therefore, in the inclusions,

$$\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \frac{r_1 - R_1}{R_1} = \frac{\sigma_1}{3K_1}. \quad (3.11)$$

Introducing (3.4) into (3.11) and the resulting expression for σ_1 into (3.10), we have

$$\sigma = C_1[1/\lambda^4 + 4/\lambda - (1/\lambda_1^4 + 4/\lambda_1)] + C_2[1/\lambda^2 - 2\lambda - (1/\lambda_1^2 - 2\lambda_1)] + 3K_1(\lambda_1 - 1). \quad (3.12)$$

Since λ_1 still obeys the relation (3.7) where c_0 is now the volume fraction of inclusions, (3.12) with (3.7) now define the stress-strain relation in isotropic deformation, while (3.12) with (3.8–3.9) define the relation between stress and volume change.

Since linear elastic inclusions are by orders of magnitude stiffer than rubbery materials, i.e. $K_1 >>> C_1, C_2$, it is quite clear that large isotropic strains cannot develop. For r_1 can only be slightly larger than R_1 since they are inclusion dimensions, and it then follows from the incompressibility relation (3.5) that r_2 can be only slightly larger than R_2. Then from (3.2) λ is only slightly larger than 1. It follows that the last term in (3.12) is dominant. It then follows from (3.4) and (3.11) that

$$\sigma \cong 3K_1\epsilon_1 = \sigma_1. \quad (3.13)$$

Thus the stress at the inclusion interface is approximately equal to the external stress.

Since the matrix is incompressible, the average volume change is now

$$\epsilon_v \cong \frac{\sigma}{3K_1} c_0, \quad (3.14)$$

from which it follows that the composite has approximately an effective linear bulk modulus given by

$$K^* = K_1/c_0. \quad (3.15)$$

This result conforms with linear elastic theory. It is recalled that K^* of the composite spheres assemblage model has been given in [7]. In the case of an incompressible linear elastic matrix, it may be shown that the result reduces to

$$K^* = \frac{1}{c}\left[K_p + \frac{4G_m}{3}(1 - c\right], \quad (3.16)$$

where c is the volume fraction of inclusions, K_p is the inclusion bulk modulus, and G_m is the matrix shear modulus. In the present case, G_m is defined for small strains by C_1 and C_2, which are negligible with respect to K_1. Neglecting the second term in (3.16), (3.15) follows.

An interesting situation develops if the interface fails in tension (dewetting). Then the inclusions suddenly become voids and the previous solution applies. Thus, in this case, the stiffness will drop precipitously by several orders of magnitude.

4. CONCLUSION

It has been shown that large deformation isotropic compression and expansion of composite materials consisting of a finite elastic matrix and spherical voids or inclusions can be exactly analyzed on the basis of the composite spheres assemblage model. Experience with this model in linear elasticity shows that its predictions are in very good agreement with experimental data.

In the present analysis it has been assumed that the matrix is an incompressible Mooney-Rivlin matrix, but it can be employed for any isotropic matrix stress-strain relation. The most interesting results obtained appear to be for voids where a marked difference in stress-strain relations in tension and compression has been demonstrated, as well as the significant effect of a minute amount of voids on extensibility.

Unfortunately, similar analysis for shear or uniaxial tension and compression is prohibitively difficult because of the loss of spherical symmetry.

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